ON REDUCING SUBSPACES FOR ONE-ELECTRON DIRAC OPERATORS*

BY

P. A. REJTÖ

ABSTRACT

A domain is specified for formal one-electron Dirac operators. It is shown presently that on this domain each of these operators admits a complete family of reducing subspaces. It is shown in the following paper that on this domain most of the physically interesting operators are essentially self-adjoint.

1. Introduction

As is well-known the Schrödinger operator corresponding to ions with one-electron admits a family of reducing subspaces. These subspaces are such that on each of them it acts like an ordinary differential operator.

In this paper we consider the case of one-electron Dirac operators. This situation is analogous but different from the previous one. To describe the main difference we need a classical result of Kato [4.b]. This says that Schrödinger operators are essentially self-adjoint on any reasonable collection of smooth functions. Hence their domain is well defined. Such a theorem for Dirac operators is available in special cases only [4.c]. Hence it is part of the mathematical analysis of these operators to define their domain.

In Section 2 we describe one-electron Dirac operators and choose a domain for them. This choice of the domain is arbitrary. Nevertheless, we have two contradictory requirements on this domain. The first one is that it should be small enough to verify the reducing subspace property. The second one is that it should be large enough to make the operator essentially self-adjoint. It is an interesting fact that both of these contradictory requirements can be satisfied for most of

* This work was supported by N.S.F. Grant GP-12361 and GB-21330

Received May 16, 1969 and in revised form June 18, 1970

the physically interesting operators. At present, however, all that we show is that for our choice of the domain the first requirement holds. Specifically in Theorem 2.1 we describe a complete family of reducing subspaces. That is to say a family of reducing subspaces such that the orthogonal sum of their closures is the entire space. In Theorem 2.2 to each subspace we assign a matrix differential operator. Then we show that the restriction of the Dirac operator to such a subspace is unitarily equivalent to the corresponding matrix-differential operator.

In Section 3 we establish Theorem 2.1. As is well known in the physics literature [3], the total angular momentum operators J_1, J_2, J_3, and the parity operator, P, commute with the Dirac operator. We use this fact to construct a family of reducing subspaces for Dirac operators. Specifically, this commuting property implies that the intersection of the eigenspaces of the operators $J_1^2 + J_2^2 + J_3^2$ and J_3 and P reduces the Dirac operator. These three operators also commute with each other and clearly they are self-adjoint. Hence the intersection of their eigenspaces forms a complete orthogonal family of subspaces. In describing the intersection of these eigenspaces we make essential use of the fact that the algebra $\{J_1, J_2, J_3\}$ defines a representation of the Lie-algebra of the three-dimensional orthogonal group. This representation is reducible and the abstract theory [2] cannot be applied to obtain its irreducible parts. Nevertheless, this algebra can be factored in such a manner that to the factor algebra the abstract theory does apply. This application is carried out in Lemmas 3.1 and 3.2. It yields a description of the intersection of the eigenspaces of $J_1^2 + J_2^2 + J_3^2$ and J_3. Combining this with an analysis of the eigenspaces of the parity operator P we arrive at the validity of Theorem 2.1.

In Section 4 we establish Theorem 2.2. In Lemma 4.1 we isolate a straightforward consequence of the commutation properties of the Dirac matrices. As to be expected the proof of Theorem 2.2 requires rather detailed calculations. These calculations are isolated in Lemmas 4.2 and 4.3. The proof of Lemma 4.2 is based on a corollary [1.b] of the Lie-algebra version [1.a] of the Wigner-Eckhart Theorem [7]. At the same time we use the fact that for the Lie-algebra of the three dimensional orthogonal group the Clebsh-Gordan and Racah coefficients have been tabulated.

It is a pleasure to thank Professors A. Galindo and P. Pascual for introducing the author to the physics literature of Dirac operators and for many valuable discussions. At the same time it is a pleasure to thank Mr. Kangel for his interest.

2. Formulation of the theorems

To describe the Dirac operator of a one-electron ion we need some notations. First let $\{g_{\mu,\nu}\}$ denote the Cartesian components of the relativistic metric tensor in suitable units. That is set,

$$g_{00} = +1, \; g_{11} = g_{22} = g_{33} = -1,$$

and

$$g_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu .$$

With the aid of these $\{g_{\mu\nu}\}$ define the Dirac matrices $\{\gamma_\mu\}$ as a set of unitary matrices satisfying the relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} . \tag{2.1$_\mu$}$$

Here and in the following Greek indices will run through the values 0, 1, 2, 3, and Latin indices will run through the values 1, 2, 3.

As is well known [3.b][9], equations $(2.1)_\mu$ with the requirement that the algebra $\{\gamma_0 \cdots \gamma_3\}$ be irreducible determine these matrices up to unitary equivalence. It follows, in particular, that they act on $\mathscr{C}_4$, the four-dimensional complex Euclidean space. In the following all that we assume is this last property and that γ_0 is self-adjoint.

Next we introduce some formal operators. These operators will be formal inasmuch as we do not specify on which subset of $\mathfrak{L}_2(\mathscr{E}_3)$ do they act.

Set

$$D_k f(x) = \frac{\partial f(x)}{\partial x_k}, \qquad x \in \mathscr{E}_3,$$

and

$$H(0) = \sum_{k=1}^{3} \gamma_0 \gamma_k \otimes \frac{1}{i} D_k + \gamma_0 \otimes I \qquad \text{in } \mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{E}_3). \tag{2.2}$$

Let

$$r(x) = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

and set

$$M\left(\frac{1}{r}\right) f(x) = \frac{1}{r(x)} f(x), \quad x \in \mathscr{E}_3, \, f \in \mathfrak{L}_2(\mathscr{E}_3).$$

In a one-electron ion with atomic number e the interaction between the electron and the nucleus is given again formally, by

$$(2.3) \qquad V(e) = -eI \otimes M\left(\frac{1}{r}\right) \quad \text{in } \mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{E}_3).$$

With the aid of these notations the total Dirac operator for such an ion can be described [3.c] as

$$(2.2)_e \qquad H(e) = H(0) + V(e).$$

Since we do not require these operators to be closed there are several different ways to specify their domains.

In the following we describe a domain which satisfies the two requirements mentioned in the introduction. To do this, we introduce some notations. Let $\mathscr{S}_2$ denote the two-dimensional unit sphere, $\mathfrak{C}_\infty(\mathscr{S}_2)$ the class of infinitely differentiable functions on it, and define $\mathfrak{L}_2(\mathscr{S}_2)$ to be the $\mathfrak{L}_2$-space with respect to the area measure on $\mathscr{S}_2$. Let the transformation T, essentially, introduce polar coordinates. Specifically, for each vector x in $\mathscr{E}_3$ set

$$\begin{aligned} x_1 &= x_1(r,\theta,\phi) = r\sin\theta\cos\phi \qquad 0 \leqq \theta < \pi \\ x_2 &= x_2(r,\theta,\phi) = r\sin\theta\sin\phi \qquad 0 \leqq \phi < 2\pi \\ x_3 &= x_3(r,\theta,\theta) = r\cos\theta. \end{aligned}$$

Then define the transformation T mapping $\mathfrak{L}_2(\mathscr{E}_3)$ onto $\overline{\mathfrak{L}_2(\mathscr{S}_2) \otimes \mathfrak{L}_2(0,\infty)}$ by

$$(2.4) \qquad Tf(r)(\theta,\phi) = rf(x_1(r\ \theta,\phi), x_2(r,\theta,\phi), x_3(r,\theta,\phi)).$$

Here and in the following the bar denotes the closure of the tensor product of two inner-product spaces.

With the aid of these notations we specify the domain of the operators in (2.2), and (2.3) to be

$$2.5) \qquad \dot{\mathfrak{D}} = (I \otimes T)^*(\mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0,\infty)),$$

Here $\dot{\mathfrak{C}}_\infty(0,\infty)$ denotes the class of infinitely differentiable complex functions on $(0,\infty)$, which vanish near zero and infinity. Following Kato [4] we use a dot to emphasize that a given operator is defined for such functions. Accordingly set,

$$(2.2)_e \qquad \dot{H}(e) = H(e) \quad \text{on } \dot{\mathfrak{D}}.$$

Next we need the notion of reducing subspace. Accordingly let A be a given, possibly unbounded operator with domain $\mathfrak{D}(A)$ in some abstract Hilbert space $\mathfrak{H}$. Suppose that a subspace $\mathfrak{S}$ of $\mathfrak{H}$ is also given. Let $P(\mathfrak{S})$ denote the ortho-projector on the range of the closure of $\mathfrak{S}$. Following Stone [12] and Kato [4] we say that the subspace $\mathfrak{S}$ reduces the operator A if

$$P(\bar{\mathfrak{S}})\mathfrak{D}(A) \subset \mathfrak{D}(A) \tag{2.6}$$

and

$$P(\bar{\mathfrak{S}})A = AP(\bar{\mathfrak{S}}) \quad \text{on } \mathfrak{D}(A). \tag{2.7}$$

In the theorem that follows we describe a complete family of reducing subspaces for this operator. That is a family of reducing subspaces whose orthogonal sum is the original space $\mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{E}_3)$. Actually it is more convenient to do this for an operator unitarily equivalent to $\dot{H}(e)$.

Definition (2.4) shows that T is an isometry if we remember the formulae connecting Cartesian and spherical coordinates. Since T is onto it is unitary. Thus we see from definition (2.5) that $\dot{H}(e)$ is unitarily equivalent to

$$(I \otimes T)\dot{H}(e)(I \otimes T)^* \quad on \quad \mathfrak{S}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0, \infty).$$

THEOREM 2.1. *For each real number e let the operator* $(I \otimes T)H(e)(I \otimes T)^*$ *be defined by equations* (2.4), $(2.2)_e$ *and* (2.5). *Then there is a complete family of ortho-projectors on* $\mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2) \otimes \mathfrak{L}_2(0, \infty)$,

$$\{P(\varepsilon, l, m),\ \varepsilon = \pm 1,\ l = 1, 2, \cdots, m = -l + \tfrac{1}{2}, -l + \tfrac{3}{2}, \cdots, l - \tfrac{1}{2}\}, \tag{2.8}$$

with the two properties that follow.

a. *To each of these projectors there is a subspace of* $\mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2)$, *say* $\mathfrak{E}(\varepsilon, l, m)$, *such that*

$$P(\varepsilon, l, m)(\mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0, \infty)) = \mathfrak{E}(\varepsilon, l, m) \otimes \dot{\mathfrak{C}}_\infty(0, \infty) \tag{2.9}$$

and

$$\mathfrak{E}(\varepsilon, l, m) \subset \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \tag{2.10}$$

and

$$\dim \mathfrak{E}(\varepsilon, l, m) = 2. \tag{2.11}$$

b. *These projectors commute with the operator* $(I \otimes T)H(e)(I \otimes T)^*$. *Specifically*

$$P(\varepsilon, l, m)(I \otimes T)\dot{H}(e)(I \oplus T)^* = (I \otimes T)\dot{H}(e)(I \otimes T)^* P(\varepsilon, l, m) \tag{2.12}$$

$$on\ \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0, \infty).$$

This theorem implies that the ranges of these projectures reduce this operator. For, setting

$$A = (I \otimes T)\dot{H}(e)(I \otimes T)^* \quad \text{and} \quad P(\mathfrak{S}) = P(\varepsilon, l, m)$$

we see from conclusions (2.9) and (2.10) that the abstract relation (2.6) holds. Similarly, we see from conclusion (2.12) the validity of the abstract relation (2.7) for these operators. That is to say, the ranges of these projectors reduce this operator as we have claimed.

Conclusion (2.11) gives additional information on these subspaces. In fact, it suggests that the part of this operator over such a reducing subspace is a two-by-two matrix differential operator. Indeed, this is the case as will be shown in the theorem that follows. To describe such a matrix-differential operator, let $\dot{M}^{-1}$ denote the restriction of M^{-1} to $\dot{\mathfrak{C}}_\infty(0,\infty)$. Using this notation, we define this operator by

$$L(e)(\kappa)\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} (-\dot{D} + \kappa\dot{M}^{-1})g + (I - e\dot{M}^{-1})f \\ (\dot{D} + \kappa\dot{M})f - (I + e\dot{M}^{-1})g \end{bmatrix},$$

(2.13)

$$\begin{bmatrix} f \\ g \end{bmatrix} \in \dot{\mathfrak{C}}_\infty((0,\infty),\mathscr{C}_2), \quad \kappa = 0, \pm 1, \pm 2, \cdots.$$

THEOREM 2.2. *There is a family of partial isometries,*

$$U(\varepsilon, l, m), \ \varepsilon = \pm 1, \ l = 1, 2, \cdots, \ m = -l + \tfrac{1}{2}, -l + \tfrac{3}{2}, \cdots, l - \tfrac{1}{2}$$

which map

(2.14) $$\mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2) \otimes \mathfrak{L}_2(0,\infty) \ \textit{onto} \ \mathfrak{L}_2((0,\infty),\mathscr{C}_2)$$

and for each ε, l, m have the three properties that follow.

a. *The initial set of $U(\varepsilon, l, m)$ is the range of the projector $P(\varepsilon, l, m)$ of Theorem 2.1. That is*

(2.15) $$U^*(\varepsilon, l, m)\, U(\varepsilon, l, m) = P(\varepsilon, l, m).$$

b. *The image of the domain of $(I \otimes T)\dot{H}(e)(I \otimes T)^*$ is contained in the domain of the operator $\dot{L}(e)\varepsilon(-1)^l l$ of definition* (2.13). *That is*

(2.16) $$U(\varepsilon, l, m) \cdot \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0,\infty) \subset \dot{\mathfrak{C}}_\infty((0,\infty),\mathscr{C}_2).$$

c. *The partial isometry* $U(\varepsilon, l, m)$ *intertwines the operator* $(I \otimes T)\dot{H}(e)(I \otimes T)$ *and* $\dot{L}(e)\varepsilon(-1)^l l$. *That is*

$$U(\varepsilon,l,m)(I \otimes T)\dot{H(e)}(I \otimes T)^* = \dot{L}(e)(\varepsilon(-1)^l l)U(\varepsilon,l,m) \quad \text{on } \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0,\infty). \tag{2.17}$$

3. **The proof of Theorem 2.1.**

We start the proof of Theorem 2.1 by exhibiting operators which commute with $\dot{H}(e)$.

A particularly simple such operator is the parity operator [3.d]. To describe it let R denote the operator of reflection with respect to the origin. Specifically set,

$$Rf(x) = f(-x), \; x \in \mathscr{E}_3, \; f \in \mathfrak{L}_2(\mathscr{E}_3).$$

Then the parity operator P is defined by

$$P = \gamma_0 \otimes R \text{ on } \mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{E}_3). \tag{3.1}$$

Since both γ_0 and R are self-adjoint isometries so is P. Hence its spectrum is given by

$$\sigma(P) = \pm 1. \tag{3.2}$$

It is an immediate consequence of definitions (2.1), (2.7) and $(2.4)_e$ that P does commute with $H(e)^{\dot{}}$, that is

$$[\dot{H(e)}, P] = 0 \quad \text{on} \quad \dot{\mathfrak{D}}. \tag{3.3}$$

To describe other operators that commute with $H(e)$ we introduce some notations. First set

$$\tau_k = \frac{i}{2}\pi(k,l,m)\gamma_e\gamma_m, \tag{3.4$_k$}$$

where (k,l,m) is any permutation of $(1,2,3)$ with parity $\pi(k,l,m)$. Next let $\dot{L}_k$ denote the infinitesimal generator of the unitary group on $\mathfrak{L}_2(\mathscr{E}_3)$, corresponding to rotations around the X_k axis of $\mathscr{E}_3$. As is well known [2.d] these operators are given by

$$\dot{L}_k = \pi(k,l,m)\frac{1}{i}(\dot{M}_l\dot{D}_m - \dot{M}_m\ddot{D}_l) \text{ on } T^*(\mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0,\infty)), \tag{3.5$_k$}$$

where

$$\dot{M}_k f(x) = x_k f(x),\ x \in \mathscr{E}_3,\ f \in T_l^*(\mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0,\infty)).$$

With the aid of these operators define the total angular momentum operators [3.f] as the Kronecker sum of $\dot{L}_k$ and τ_k. That is set

$$(3.6)_k \qquad \dot{J}_k = I \otimes \dot{L}_k + \tau_k \otimes I \quad \text{on} \quad \dot{\mathfrak{D}}.$$

We maintain that each of these operators commutes with $\dot{H}(e)$. In other words,

$$(3.7)_k \qquad [\dot{H}(e), \dot{J}_k] = 0 \quad \text{on} \quad \dot{\mathfrak{D}}.$$

For, definitions $(3.6)_1$ and $(2.2)_e$ yield

$$(3.8) \qquad [\dot{H}(e), \dot{J}_1] = \gamma_0\gamma_2 \otimes \left[\dot{L}_1, \frac{1}{i}D_2\right] + \gamma_0\gamma_3 \otimes \left[\dot{L}_1, \frac{1}{i}D_3\right]$$
$$+ [\tau_1, \gamma_0\gamma_2] \otimes \frac{1}{i}D_1 + [\tau_1, \gamma_0\gamma_3] \otimes \frac{1}{i}D_3 + [\tau_1, \gamma_0] \otimes I.$$

Elementary algebra shows that definitions $(3.5)_k$ imply

$$\left[L_1, \frac{1}{i}D_k\right] = \begin{cases} 0 & k = 1 \\ D_{k+1} & k = 2 \\ -D_{k-1} & k = 3. \end{cases}$$

Similarly, definitions (2.1) and (3.1), imply

$$[\tau_1, \gamma_0\gamma_k] = \begin{cases} 0 & k = 1 \\ \gamma_0\gamma_{k+1} & k = 2 \\ -\gamma_0\gamma_{k-1} & k = 3. \end{cases}$$

Inserting these two relations in equation (3.8) we obtain the validity of relation $(3.7)_1$. A similar argument, that we shall not carry out yields the validity of all of relations $(3.7)_k$.

We see from these relations that $\dot{H}(e)$ commutes with each member of the algebra $\{\dot{J}_1, \dot{J}_2, \dot{J}_3, P\}$. In particular it commutes with each of the three operators $\dot{J}_1^2 + \dot{J}_2^2 + \dot{J}_3^2, \dot{J}_3$ and P. According to the arguments that follow these three operators commute pairwise and the spectra of their closures is discrete. Since these closures are self-adjoint the intersection of their eigenspaces forms a complete family of orthogonal subspaces. At the same time it follows that each of these intersections reduces the operator $\dot{H}(e)$.

To study these eigenspaces first we need the well known fact that these operators act, essentially, in $\mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2)$. This is described in more specific terms in the lemma that follows.

LEMMA 3.1. *Let the transformation T be defined by equation* (2.4). *Then to the operators $\dot{J}_k$ and P of definitions* $(3.6)_k$ *and* (3.1) *there are operators $\dot{S}_k$ and Q acting in $\mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2)$ such that*

$$(I \otimes T)\dot{J}_k(I \otimes T)^* = \dot{S}_k \otimes I \tag{3.9$_k$}$$

and *on* $\mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0,\infty)$

$$(I \otimes T)P(I \otimes T)^* = (\gamma_0 \otimes Q) \otimes I. \tag{3.10}$$

To verify conclusions $(3.9)_k$ we need that [2.c] there are operators $\dot{A}_k$ such that

$$T\dot{L}_k T^* = \dot{A}_k \otimes I \ \ on \ \ \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0,\infty). \tag{3.11$_k$}$$

Hence setting

$$\dot{S}_k = \tau_k \otimes I + I \otimes \dot{A}_k \ \ on \ \ \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0,\infty) \tag{3.12$_k$}$$

we obtain the validity of conclusions $(3.9)_k$. To verify conclusion (3.10) let the operator Q on $\mathfrak{L}_2(\mathscr{S}_2)$ be defined by

$$Qf(\theta,\phi) = f(\pi - \theta, \phi + \pi). \tag{3.13}$$

Then it is an elementary fact that

$$TRT^* = Q. \tag{3.14}$$

Hence remembering definition (3.1) we obtain the validity of conclusion (3.10). This completes the proof of Lemma 3.1.

Let S_k denote the closure of the operators $\dot{S}_k$ of Lemma 3.1. We observe that these operators are such that $S_1^2 + S_2^2 + S_3^2$, S_3 and $\gamma_0 \otimes Q$ commute pairwise. That is to say*

$$[S_1^2 + S_2^2 + S_3^2, S_3] = 0, \tag{3.15}$$

and

$$[S_3, \gamma_0 \otimes Q] = 0 \ \ and \ \ [S_1^2 + S_2^2 + S_3^2, \gamma_0 \otimes Q] = 0. \tag{3.16}$$

---

* Actually these three operators generate a maximal Abelian subalgebra, however, at present we shall not be concerned with this fact.

To verify relation (3.15) recall definitions $(3.4)_k$ and $(3.11)_k$. They yield the wel known relations

$$(3.17)_k \qquad [\tau_k, \tau_l] = i\pi(k,l,m)\tau_m, \text{ and}$$

$$(3.18)_k \qquad [A_k, A_l] = i\pi(k,l,m)A_m.$$

These two relations, in turn, together with definitions $(3.12)_k$ yield

$$(3.19)_k \qquad [S_k, S_l] = i\pi(k,l,m)S_m.$$

Inserting this relation in

$$[S_1^2 + S_2^2 + S_3^2, S_3] = \sum_{k=1}^{2} ([S_k, S_3]S_k + S_k[S_k, S_3])$$

we obtain the validity of relation (3.15). To verify relations (3.16) recall definitions $(2.1)_\mu$. They, together with definitions $(3.4)_k$ show that

$$(3.20)_k \qquad \tau_k\gamma_0 = \gamma_0\tau_k.$$

At the same time we see that

$$RL_k = L_kR \qquad on \quad \dot{\mathfrak{C}}_\infty(\mathscr{E}_3).$$

Hence remembering relation (3.14) and definitions (3.11_k) we obtain

$$A_kQ = QA_k \qquad on \quad \mathfrak{C}_\infty(\mathscr{S}_2).$$

Combining these relations with $(3.18)_k$ we arrive at

$$(3.21)_k \qquad [S_k, \gamma_0 \otimes Q] = 0.$$

From these relations, in turn, we arrive at the validity of the two equations in (3.16).

According to relations (3.15) and (3.16) the operators $S_1^2 + S_2^2 + S_3^2$, S_3, and $\gamma_0 \otimes Q$ have common eigenspaces. The lemma that follows describes these eigenspaces. In it we call the restriction of an operator to a reducing subspace the part of the operator over the subspace. For a given self-adjoint operator A we denote by $E(A)(\cdot)$ the spectral family of A.

LEMMA 3.2. *The operators of Lemma* 3.1 *are such that*

$$(3.22) \qquad \sigma(\gamma_0 \otimes Q) = \{-1, +1\}$$

and

$$(3.23) \qquad \sigma(S_1^2 + S_2^2 + S_3^2) = \{(j+1)j,\ j = \tfrac{1}{2}, \tfrac{3}{2}, \cdots\}.$$

Furthermore for each such j and $\varepsilon = \pm 1$ *the spectrum of the part of* S_3 *over the corresponding eigenspace is given by*

(3.24) $$\sigma(S_3 \cdot E(\gamma_0 \otimes Q)(\varepsilon)E(S_1^2 + S_2^2 + S_3^2)(j+1)j) = \{-j, -j+1, \cdots +j\},$$

and for each half-integer m in $[-j, +j]$

(3.25) $$\dim\{E(S_3)(m)E(\gamma_0 \otimes Q)(\varepsilon)E(S_1^2 + S_2^2 + S_3^2)((j+1)j) \cdot \mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2)\} = 2.$$

To verify conclusion (3.22) recall defiflitions $(2.1)_0$ and (3.13). They show that

$$\gamma_0^2 = I \quad on \ \mathscr{C}_4 \quad and \quad Q^2 = I \quad on \ \mathfrak{L}_2(\mathscr{S}_2).$$

Hence

(3.26) $$\sigma(\gamma_0) = \sigma(Q) = \{-1, +1\}$$

This fact together with the well known fact that the spectrum of a Kronecker product equals the product of the spectra establishes the validity of conclusion (3.22), [14] [15].

To verify conclusion (3.23) recall relations $(3.17)_k$ and $(3.18)_k$. They show that each of the algebras $\{\tau_1, \tau_2, \tau_3\}$ and $\{A_1, A_2, A_3\}$ is isomorphic to a Lie algebra of the three dimensional rotation group. Hence according to definitions $(3.12)_k$ the algebra $\{S_1, S_2, S_3\}$ is the Kronecker product of two such algebras. It is an elementary fact that if the algebra $\{\tau_1, \tau_2, \tau_3\}$ is irreducible on $\mathfrak{E}(\tau)$ and the algebra $\{A_1, A_2, A_3\}$ is irreducible on $\mathfrak{E}(A)$ then the algebra $\{S_1, S_2, S_3\}$ is reducible on $\mathfrak{E}(\tau) \otimes \mathfrak{E}(A)$. A well known abstract theorem [2.b] describes the irreducible subspaces of $\mathfrak{E}(\tau) \otimes \mathfrak{E}(A)$. At the same time it describes the common eigenspaces of the parts of $S_1^2 + S_2^2 + S_3^2$ and S_3 over $\mathfrak{E}(\tau) \otimes \mathfrak{E}(A)$.

The subspaces of $\mathfrak{C}_\infty(\mathscr{S}_2)$ on which the algebra $\{A_1, A_2, A_3\}$ is irreducible are well known [2.e]. In fact they are eigenspaces of the operator $A_1^2 + A_2^2 + A_3^2$. At the same time we know that

(3.27) $$\sigma(A_1^2 + A_2^2 + A_3^2) = \{(l+1)l, l = 0, 1, \cdots\}.$$

For brevity we denote by $A_k((l+1)l)$ the corresponding part of the operator A_k, that is, set

$$A_k((l+1)l) = A_k E(A_k)((l+1)l).$$

The subspaces of $\mathscr{C}_4$ on which the algebra $\{\tau_1, \tau_2, \tau_3\}$ is irreducible are easy to describe. For, we see from relations $(3.20)_k$ and (3.26) that each of the two subspaces

$$(3.28)_\delta \qquad \mathscr{C}(\delta) = E(\gamma_0)(\delta)\mathscr{C}_4$$

reduces this algebra. That is to say

$$\tau_k\mathscr{C}(\delta) \subset \mathscr{C}(\delta).$$

Since each τ_k is invertible this inclusion is actually an equality. From this fact and from the fact that the algebra $\{\tau_1, \tau_2, \tau_3\}$ is not commutative we obtain

$$\dim \mathscr{C}(\delta) = 2.$$

From this, in turn, we obtain that this algebra is irreducible on each of the two subspaces $\mathscr{C}(\delta)$.

For brevity set,

$$(3.29)_l \qquad S_k(\delta,(l+1)l) = S_k(\mathscr{C}(\delta) \otimes E(A_1^2 + A_2^2 + A_3^2)(l(+1)l) \cdot \mathfrak{L}_2(\mathscr{S}_2),$$

where the right member denotes the part of the operator S_k over the reducing subspace in the bracket.

After these preparations we can apply the previously mentioned abstract theorem [2.b] to the operators in $(3.29)_l$. This yields for each positive integer l.

$$(3.30)_l \qquad \sigma(S_1^2 + S_2^2 + S_3^2)(\delta,(l+1)l) = \{(l+\tfrac{1}{2})(l+\tfrac{3}{2}),\ \left|l-\tfrac{1}{2}\right|(\left|l-\tfrac{1}{2}\right|+1)\}.$$

Note that the value $l = 0$ is exceptional inasmuch as in this case the spectrum consists of a single point. Remembering relation (3.27) we see from the spectral theorem [4.2] that

$$(3.31) \qquad (\mathscr{C}(+1) \oplus \mathscr{C}(-1) \otimes \left(\sum_{l=0}^{\infty} E(A_1^2 + A_2^2 + A_3^2)((l+1)l)\right)\mathfrak{L}_2(\mathscr{S}_2) = \mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2).$$

Hence each of the operators S_k is the orthogonal sum of the operators $S_k(\delta,(l+1)l)$. As is well known, the spectrum of an orthogonal sum is the union of the spectra of the terms. Inserting this fact in relations $(3.30)_l$ we arrive at the validity of conclusion (3.23).

To verify conclusion (3.24) we use again the abstract theorem [2.b]. This shows that for each $\delta = \pm 1$ and for positive integer l and for

$$j = l + \tfrac{1}{2} \quad \text{or} \quad j = \left|l - \tfrac{1}{2}\right|$$

we have

$$(3.32)_l \qquad \sigma(S_3 \cdot E(S_1^2 + S_2^2 + S_3^2)(\delta,(l+1)l)((j+1)j) = \{-j, -j+1, \cdots + j\}.$$

At the same time it follows that each of these eigenvalues is simple. We shall make essential use of this fact in the proof of conclusion (3.25). Combining relations $(3.32)_l$ with definitions $(3.29)_l$ we arrive at the validity of conclusion (3.24).

To verify conclusion (3.25) set

(3.33) $\quad j(l) = l - \frac{1}{2},\ l = 1,2,\cdots \ \ \textit{and} \ \ l(j) = j + \frac{1}{2},\ j = \frac{1}{2},\ \frac{3}{2},\cdots$

Remembering relations $(3.30)_l$ we see from this definition that for each j,

$$E(S_1^2 + S_2^2 + S_3^2)((j+1)j) = \sum_{\delta=-1}^{+1} E(S_1^2 + S_2^2 + S_3^2)(\delta,(l(j)-1)l(j))((j+1)j)$$

$$+ \sum_{\delta=-1}^{+1} E(S_1^2 + S_2^2 + S_3^2)(\delta,(l(j)+1)l(j))((j+1)j).$$

Since according to relation (3.15) the operators $S_1^2 + S_2^2 + S_3^2$ and S_3 commute lwe obtain

$$E(S_3)(m) \cdot E(S_1^2 + S_2^2 + S_3^2)((j+1)j)$$

(3.34)

$$= \sum_{\delta=-1}^{+1} (S_3(\delta,(l(j)+1)l(j)))(m) \cdot E(S_1^2 + S_2^2 + S_3^2)(\delta,(l(j)+1)l(j)))((j+1)j)$$

$$+ \sum_{\delta=-1}^{+1} E(S_3(\delta,(l(j)-1)l(j))(m) \cdot E(S_1^2 + S_2^2 + S_3^2)(\delta,(l(j)+1)l(j)))((j+1)j).$$

We maintain that for each $\delta = \pm 1$,

$(3.35)_\delta \quad \gamma_0 \otimes Q = \delta(-1)^l I \ \ on \ \ \mathscr{C}(\delta) \otimes E(A_1^2 + A_2^2 + A_3^2)((l+1)l) \cdot \mathfrak{L}_2(\mathscr{S}_2).$

As is well known [2.f] these eigenspaces are spanned by spherical harmonics. Specifically let $y(l,m)$ denote the spherical harmonics in the usual notation [2.e], which is also given in equation (I.3) of the Appendix. Then

(3.36) $\quad E(A_1^2 + A_2^2 + A_3^2) \cdot \mathfrak{L}_2(\mathscr{S}_2) = \{y(l,-l),\cdots y(l,+l)\}.$

At the same time it follows [3.e] that

$(3.37)_l \quad Qy(l,m) = (-1)^l y(l,m).$

These two facts together with definition $(3.28)_\delta$ yield the validity of relation $(3.35)_\delta$. Applying this relation to the value $\delta = \varepsilon(-1)^l$ we see from definitions $(3.30)_l$ that

$$E(\gamma_0 \otimes Q)(\varepsilon) \cdot \sum_{\delta=-1}^{+1} E(S_3(\delta, l+1)l))(m) = E(S_3(\varepsilon(-1)^l,(l+1)l))(m),$$

if we remember that in view of the self-adjointness of $\gamma_0 \otimes Q$ the eigen-projectors are orthogonal. Similarly we see that

$$E(\gamma_0 \otimes Q) \otimes \sum_{\delta=-1}^{+1} E(S_3(\delta,(l-1)l))(m) = E(S_3(\varepsilon(-1)^{l-1},(l-1)l))(m).$$

Inserting these two equations in relation (3.34) we arrive at

$$\begin{aligned} &E(\gamma_0 \otimes Q)(\varepsilon) \cdot E(S_1^2 + S_2^2 + S_3^2)((j+1)j) \cdot E(S_3)(m) \\ (3.38) \quad &= E(S_1^2 + S_2^2 + S_3^2)(\varepsilon(-1)^{l(j)}, (l(j)+1)l(j))((j+1)j) \cdot E(S_3)(m) \\ &+ E(S_1^2 + S_2^2 + S_3^2)(\varepsilon(-1)^{l(j)-1}, l(j)(l(j)-1))((j+1)j) \cdot E(S_3)(m), \end{aligned}$$

if we remember relation (3.15) which says that the operators $S_1^2 + S_2^2 + S_3^2$ and S_3 commute. According to relations $(3.32)_l$ the rank of each of the two projectors on the right of (3.38) equals one. Hence the rank of the projector on the left equals two, which establishes conclusion (3.25). This, in turn, establishes Lemma 3.2.

Finally we derive Theorem 2.1 from these two lemmas.

To verify conclusion (2.8), for each triplet (ε, l, m), where $\varepsilon = \pm 1$, l is a positive integer and m is a half-integer in $[-j(l), +j(l)]$ we set

$$(3.39) \qquad P(\varepsilon,l,m) = (I \otimes T) \cdot E(P)(\varepsilon)(J_1^2 + J_2^2 + J_3^2)((j(l)+1)j(l)) \cdot \\ \cdot E(J_3)(m) \cdot (I \otimes T)^*.$$

Relations (3.15) and (3.16) together with conclusions $(3.9)_k$ and (3.10) of Lemma 3.1 show that the three operators $J_1^2 + J_2^2 + J_3^2$, J_3 and P commute pairwise. Hence the product of the three ortho-projectors on the right of (3.39) is again an ortho-projector [6]. Remembering the spectral theorem [4.a] we see from conclusions (3.22), (3.23) and (3.24) of Lemma 3.2 and from definition (3.33) that the sum of these ortho-projectors equals the identity operator. That is to say conclusion (2.8) holds.

To verify conclusion (2.9) of Theorem 2.1 insert definition (3.39) in Lemma 3.1. This yields

$$(3.40) \qquad \begin{aligned} P(\varepsilon,l,m) &= E(\gamma_0 \otimes Q)(\varepsilon) \cdot E(S_1^2 + S_2^2 + S_3^2)((j(l)+1)j(l)) \cdot E(S_3)(m) \otimes I \\ &\quad on\ \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \mathfrak{L}_2(0,\infty). \end{aligned}$$

Hence setting

$$E(\varepsilon, l, m)$$

(3.41)

$$= E(\gamma_0 \otimes Q)(\varepsilon) \cdot E(S_1^2 + S_2^2 + S_3^2)((j(l)+1)j(l))E(S_3)(m) \cdot (\mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2))$$

we obtain the validity of conclusion (2.9).

To verify conclusion (2.10) recall relations $(3.30)_l$ and definition (3.33). They show that for each positive half-integer j

$$E(S_1^2 + S_2^2 + S_3^2)((j+1)j) \cdot \mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2) \subset$$

$$\subset \mathscr{C}_4 \otimes \{E(A_1^2 + A_2^2 + A_3^2)((l(j)-1)l(j)) + E(A_1^2 + A_2^2 + A_3^2)((l(j)+1)l(j))\}\mathfrak{L}_2(\mathscr{S}_2).$$

According to relation (3.36) each of these subspaces are spanned by spherical harmonics. Using that the spherical harmonics are smooth functions we obtain the validity of conclusion (2.10).

Conclusion (2.11) of Theorem 2.1 is an immediate consequence of conclusion (3.25) of Lemma 3.2 and of definition (3.41).

To verify conclusion (2.12) recall the already established conclusions (2.9) and (2.10). They show that

$$P(\varepsilon, l, m)(\mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0, \infty)) \subset \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0, \infty). \tag{3.42}$$

Definitions (3.1) and $(3.6)_k$ show that each of the operators

$$(I \otimes T)P(I \otimes T)^* \quad and \quad (I \otimes T)\, J_k(I \otimes T)^*$$

maps the set on the right of (3.42) into itself. Hence remembering the commutator relation $(3.7)_3$ we see that

$$f \in \mathscr{C}_4 \otimes \dot{\mathfrak{C}}_\infty(0, \infty) \quad and \quad (I \otimes T)J_3(I \otimes T)^* f = mf$$

implies

$$(I \otimes T)J_3 \dot{H}(e)(I \otimes T)^* f = m(I \otimes T)\dot{H}(e)(I \otimes T)^* f.$$

Thus for each m,

$$(I \otimes T)E(J_3)(m)\dot{H}(e)\,(I \otimes T)^* = (I \otimes T)\dot{H}(e)E(J_3)(m)(I \otimes T)^*$$

$$on \ \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0, \infty).$$

At the same time we see that analogous relations hold for the projectors $E(J_1^2 + J_2^2 + J_3^2)((j+1)j)$ and $E(P)(\varepsilon)$. Inserting these relations in definition (3.40) we arrive at the validity of conclusion (2.12).

4. The proof of Theorem 2.2.

We start the proof of Theorem 2.2 with a lemma. In it, as before, $\dot{M}$ denotes the multiplication operator on $\dot{\mathfrak{C}}_\infty(0,\infty)$ and $\dot{M}^{-1}$ denotes the restriction of M^{-1} to this set. Similarly, $\dot{D}$ denotes differentiation on this set.

LEMMA 4.1. *There are operators B and C acting on $\mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2)$ such that*

$$(4.1)\qquad \begin{aligned} &(I \otimes T)\cdot \sum_{k=1}^{3} \left(\gamma_0\gamma_k \otimes \frac{1}{i}\dot{D}_k\right)\cdot (I\otimes T)^* = B\otimes \frac{1}{i}(\dot{D}-\dot{M}^{-1}) + C\otimes \dot{M}^{-1} \\ &\qquad\qquad on\ (\mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2)\otimes \dot{\mathfrak{C}}_\infty(0,\infty)). \end{aligned}$$

To describe such operators B and C we first define a matrix valued function of the variable x in $\mathscr{E}_3$ by setting

$$(4.2)\qquad \gamma_r(x) = \frac{1}{r(x)} \sum_{k=1}^{3} x_k\gamma_k .$$

Definitions (2.2) and (2.3) show that for every non-zero vector x in $\mathscr{E}_3$

$$\gamma_r(x) = \gamma_r\left(\frac{x}{|x|}\right).$$

This in turn, shows that there is an operator B on $\mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2)$ such that

$$(4.3)\qquad -(I\otimes T)\cdot M(\gamma_r\gamma_0)\cdot (I\otimes T)^* = B\otimes I \ \ on\ \mathscr{C}_4\otimes\mathfrak{C}_\infty(\mathscr{S}_2)\otimes\dot{\mathfrak{C}}_\infty(0,\infty).$$

With the aid of this operator B and definitions $(3.4)_k$ and $(3.12)_k$ we define the operator C on $\mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2)$ by setting

$$(4.4)\qquad C = 2iB \cdot \sum_{k=1}^{3} \tau_k \otimes A_k .$$

We maintain that conclusion (4.1) holds with reference the operators of definitions (4.3) and (4.4). For, we see from definition (2.1) that

$$\gamma_0\gamma_l\gamma_0\gamma_m = \begin{cases} -\gamma_l\gamma_m & l \neq m \\ I & l = m. \end{cases}$$

Similarly, we see from definition $(3.4)_k$ that if (k,l,m) is a given permutation of $(1,2,3)$ with parity $\pi(k,l,m)$ then

$$-\gamma_l\gamma_m = \frac{1}{\pi(k,l,m)} 2i\tau_k .$$

These two relations together with definition $(3.5)_k$ yield

$$\left(\sum_{k=1}^{3} \gamma_0\gamma_k \otimes \dot{M}_k\right) \cdot \left(\sum_{k=1}^{3} \gamma_0\gamma_k \otimes \frac{1}{i}\dot{D}_k\right) = I \otimes \sum_{k=1}^{3} \dot{M}_k \frac{1}{i}\dot{D}_k + 2i \sum_{k=1}^{3} \tau_k \otimes \dot{L}_k \tag{4.5}$$

$$on\ (I \otimes T)^*(\mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0, \infty)).$$

Definitions (4.2) and (2.1) show that

$$\gamma_r^{-1}(x) = -\gamma_r(x).$$

This, in turn, shows that

$$(\gamma_1 \otimes \dot{M}_1 + \dot{\gamma}_2 \otimes \dot{M}_2 + \gamma_3 \otimes \dot{M}_3)^{-1} = -\dot{M}\left(\frac{1}{r}\gamma_r\right)$$

and from definition (2.1) we have

$$\gamma_0^{-1} = \gamma_0.$$

Inserting the last two equations in (4.5) we obtain

$$\sum_{k=1}^{3} \left(\gamma_0\gamma_k \otimes \frac{1}{i}\dot{D}_k\right) = - M(\gamma_r\gamma_0) \otimes \dot{M}\left(\frac{1}{r}\right) \sum_{k=1}^{3} \left(\dot{M}_k \frac{1}{i}\dot{D}_k\right)$$

$$- 2iM(\gamma_r\gamma_0) \otimes \dot{M}\left(\frac{1}{r}\right) \cdot \sum_{k=1}^{3} \tau_k \otimes \dot{L}_k \tag{4.6}$$

$$on\ \mathscr{C}_4 \otimes T^*(\mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0, \infty)).$$

Clearly

$$\sum_{k=1}^{3} x_k \frac{\partial}{\partial x_k} \left(\frac{1}{r(x)}\right) = -\frac{1}{r(x)},$$

and

$$T\dot{M}\left(\frac{1}{r}\right)T^* = \dot{M}^{-1}. \tag{4.7}$$

These two relations together with the chain rule and the Euler formula for homogeneous functions yield

$$T\dot{M}\left(\frac{1}{r}\right) \left(\sum_{k=1}^{3} \dot{M}_k \frac{1}{i}\dot{D}_k\right)T^* = I \otimes \frac{1}{i}(\dot{D} - \dot{M}^{-1})\ on\ \dot{\mathfrak{C}}_\infty(\mathscr{S}_2) \otimes \mathfrak{C}_\infty(0, \infty). \tag{4.8}$$

Taking the Kronecker product of this equation with the identity operator on $\mathscr{C}_4$ and multiplying the result equation by equation (4.3) we obtain,

$$-(I\otimes T)\cdot M(\gamma_r\gamma_0)\otimes \dot{M}\left(\frac{1}{r}\right)\ \Sigma\ \left(\dot{M}_k\frac{1}{i}\dot{D}_k\right)\cdot(I\otimes T)^* = B\otimes\frac{1}{i}(\dot{D}-\dot{M})^{-1} \tag{4.9}$$

$$on\ (\mathscr{C}_4\otimes C_\infty(\mathscr{S}_2))\otimes\dot{\mathfrak{C}}_\infty(0,\infty).$$

We see from equation (4.7) and definition $(3.12)_k$ that

$$(I\otimes T)\cdot\dot{M}\left(\frac{1}{r}\right)\left(\sum_{k=1}^{3}\tau_k\otimes\dot{L}_k\right)\cdot(I\otimes T)^*=\left(\sum_{k=1}^{3}\tau_k\otimes A_k\right)\otimes\dot{M}^{-1}$$

$$on\ \mathscr{C}_4\otimes\mathfrak{C}_\infty(\mathscr{S}_2)\otimes\mathfrak{C}_\infty(0,\infty).$$

Multiplying this equation by $2i$-times equation (4.3) we obtain

$$-(I\otimes T)\cdot M(\gamma_r\gamma_0)\otimes\dot{M}\ \left(\frac{1}{r}\right)\cdot\left(2i\sum_{k=1}^{3}\tau_k\otimes\dot{L}_k\right)\cdot(I\otimes T)^* = C\otimes\dot{M}^{-1} \tag{4.10}$$

$$on\ \mathscr{C}_4\otimes\mathfrak{C}_\infty(\mathscr{S}_2)\otimes\dot{\mathfrak{C}}_\infty(0,\infty),$$

if we remember definition (4.4) and that T is unitary. Finally inserting equation (4.10) and (4.9) in equation (4.6) we arrive at the validity of conclusion (4.1). This completes the proof of Lemma 4.1.

We shall also need a more specific description of the operators B and C of Lemma 4.1. This is done in the two lemmas that follow. To formulate these lemmas recall equation (4.31) and the fact that the projectors on the right of this equation are of rank one. Hence for each ε, j, and half integer m in $[-j,+j]$ we can define two unit vectors such that

$$\{y(\varepsilon,(j+1)j,(l(j)\pm\varepsilon)l(j),m)\} =$$

$$(4.11)_\pm$$

$$= E(\gamma_0\otimes Q)(\varepsilon)\cdot E(S_3)(m)\cdot E(S_1^2+S_2^2+S_3^2)(\varepsilon(-1)^{l(j)},l(j)\pm1)l(j))((j+1)j)\cdot$$

$$\cdot\,\mathscr{C}_4\otimes\mathfrak{L}_2(\mathscr{S}_2),$$

where the function $l(j)$ is defined by (3.33). Relations (3.38), (3.24) and (3.25) imply that these vectors form an orthonormal basis in $\mathscr{C}_4\otimes\mathfrak{L}_2(\mathscr{S}_2)$.

Lemma 4.2. *Let $\varepsilon=\pm1$. Then the action of the operator B of definition* (4.3) *on the basis vectors of* $(4.11)_\pm$ *is given by*

$$(4.12)_{\pm1}\quad By(\varepsilon,(j+1)j,(l(j)\pm1)l(j),m) = -y\,(\varepsilon,(j+1)j,(l(j)\mp1)l(j),m).$$

To verify conclusions $(4.12)_{\pm1}$ we need an elementary fact concerning Dirac matrices. Specifically we need that there is an isometry ω on $\mathscr{C}_4$ such that for each k,

$(4.13)_k$ $$\gamma_0\gamma_k = \omega(2\tau_k)$$

and

$(4.14)_k$ $$\omega\tau_k = \tau_k\omega .$$

In analogy to definition (4.2) for each vector x in $\mathscr{E}_3$ set

(4.15) $$\tau_r(x) = \frac{1}{r(x)} \sum_{k=1}^{3} x_k\tau_k .$$

Then in analogy to definition (4.3) we can define an operator F by the requirement that

(4.16) $$(I \otimes T)M(\tau_r)(I \otimes T)^* = F \otimes I \quad on \quad \mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2) \otimes \mathfrak{L}_2(0,\infty).$$

Remembering relations $(4.13)_k$, we see from this definition and from (4.3) that

(4.17) $$B = (\omega \otimes I)2F \quad on \quad \mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2).$$

It is clear from relations $(4.12)_k$ that

$$S_k(\omega \otimes I) = (\omega \otimes I)S_k \quad on \quad \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2),$$

and evidently

$$(I \otimes A_k)(\omega \otimes I) = (\omega \otimes I)(I \otimes A_k) \quad on \quad \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2).$$

It is an elementary consequence of relations $(4.13)_k$ that

$$\gamma_0\omega = -\omega\gamma_0 \quad hence \quad (\gamma_0 \otimes Q)(\omega \otimes I) = -(\omega \otimes I)(\gamma_0 \otimes Q).$$

These three relations together show that

$(4.18)_{\pm 1}$ $$(\omega \otimes I)y(\varepsilon,(j+1)j,(l \pm 1)l,m) = y(-\varepsilon,(j+1)j,(l \pm 1)l,m),$$

if we remember definition $(4.11)_\pm$ and that ω is an isometry. These two relations, together with relation (4.17) show that conclusions $(4.12)_{\pm 1}$ are implied by

$(4.19)_{\pm 1}$ $$Fy(\varepsilon,(j+1)j,(l(j) \pm 1)l(j),m) = \tfrac{1}{2}y(-\varepsilon,(j+1)j,(l(j) \mp 1)l(j),m).$$

To verify relations $(4.19)_{\pm 1}$ recall definition (4.13) which shows that

$$\tau_r(x) = \frac{x_3}{r(x)}\tau_3 + \frac{x_1 - ix_2}{r(x)}\,\frac{\tau_1 + i\tau_2}{2} + \frac{x_1 + ix_2}{r(x)}\,\frac{\tau_1 - i\tau_2}{2}.$$

As is well known [1.e] the homogeneous functions on the right are closely related to spherical harmonics. Specifically, we have

$$\frac{x_3}{r(x)}(\theta,\phi) = \sqrt{\frac{4\pi}{3}}y(1,0)(\theta,\phi), \frac{x_1 \pm ix_2}{r(x)}(\theta,\phi) = \mp\sqrt{2}\cdot\sqrt{\frac{4\pi}{3}}y(1,\pm 1)(\theta,\phi).$$

Next set

$$\tau(1,0) = \tau_3, \ \tau(1,\pm 1) = \frac{\pm 1}{\sqrt{2}}(\tau_1 \pm i\tau_2).$$

Insertion of these relations in definitions (4.15) and (4.16) yields

$$F = \sqrt{\frac{4\pi}{3}} \sum_{m=-1}^{+1} (-1)^m \tau(1,m) \otimes M(y(1,-m)). \tag{4.20}$$

At the same time we see that

$$\begin{aligned} [\tau_3, \tau(1,m)] &= m\tau(1,m) \\ [\tau_1 \pm i\tau_2, \tau(1,m)] &= \sqrt{(1 \mp m)(1 \pm m + 1)}\tau(1,m) \\ m &= -1, 0, 1. \end{aligned} \tag{4.21}$$

We have seen in Section 3 that the algebra $\{\tau_1, \tau_2, \tau_3\}$ is irreducible on $\mathscr{C}(\pm 1)$. Let $\tau_k(\pm 1)$ denote the restriction of τ_k to $\mathscr{C}(\pm 1)$. Clearly the operators $\tau_k(+1)$ and $\tau_k(-1)$ satisfy the commutator relations $(3.14)_k$. Hence we have a canonical basis for each of these two sets of operators. That is to say we have vectors in $\mathscr{C}_4$ such that

$$\{c(\pm 1, \tfrac{1}{2}, +\tfrac{1}{2}), c(\pm 1, \tfrac{1}{2}, -\tfrac{1}{2})\} = \mathscr{C}(\pm 1)$$

and

$$[\tau_1(\pm 1) \pm i\tau_2(\pm 1)]c(\pm 1, \tfrac{1}{2}, +\tfrac{1}{2}) = \sqrt{(\tfrac{1}{2} \pm \tfrac{1}{2})(\tfrac{1}{2} \pm \tfrac{1}{2} + 1)} \ c(\pm 1, \tfrac{1}{2}, -\tfrac{1}{2})$$

$$\tau_3(\pm 1)c(\pm 1, \tfrac{1}{2}, \pm\tfrac{1}{2}) = \pm\tfrac{1}{2} c(\pm 1, \tfrac{1}{2}, \pm\tfrac{1}{2}).$$

Relations (4.21) allow us to apply to the operators $\tau(\pm 1)(1, m)$ Racah's version of the Wigner-Eckhart [1.a]. This theorem describes the matrix elements of the operators $\tau(\pm 1)(1,m)$ with reference to the canonical basis. Specifically it says that there is a complex number, which we denote by $(\tfrac{1}{2}, \| \tau(\pm 1)(1) \|, \tfrac{1}{2})$, such that for every admissible m_1, m_2, m we have

$$\begin{aligned} (c(\pm 1, \tfrac{1}{2}, m_2), \tau(\pm 1)(1,m)c(\pm, \tfrac{1}{2}, m_1)) \\ = C(\tfrac{1}{2}, \tfrac{1}{2}, 1, m_1, m_2, m) \cdot (\tfrac{1}{2}, \| \tau(\pm 1)(1) \|, \tfrac{1}{2}) \end{aligned} \tag{4.22}$$

Here the first factor is a Clebsh-Gordan coefficient in the notation of [1].

As is well known [2.e], for each l the spherical harmonics $y(l,m)$ form a canonical basis for an appropriate restriction of the operators A_k of definition. $(3.12)_k$. Specifically the algebra $\{A_1, A_2, A_3\}$ is irreducible on the subspace

$$\{y(l,m), \ m = -l, \cdots, 0, \cdots, +l\},$$

and

$$(A_1^2 + A_2^2 + A_3^2)y(l,m) = l(l+1)y(l,m)$$
$$A_3 y(l,m) = my(l,m).$$

Since

$$y(l,m) \in \mathfrak{C}_\infty(\mathscr{S}_2)$$

and the operators A_k are differential operators, we have

$$[A_k, M(y(l,m))] = M(A_k y(l,m)) \text{ on } \mathfrak{C}_\infty(\mathscr{S}_2).$$

Since for each l the spherical harmonics form a canonical basis, we obtain [2.g]

$$(4.23) \qquad \begin{aligned} &[A_3, M(y(l,m))] = mM(y(l,m)), \\ &[A_1 \pm iA_2, M(y(l,m))] = \sqrt{l(\mp m)(l \pm m+1)} \cdot M(y(l, m \pm 1)), \\ &m = -l, \cdots, 0, \cdots, \ l. \end{aligned}$$

These relations allow us to apply to the operators $M(y(l,m))$ the previously mentioned version [1.a] of the Wigner-Eckhart theorem. This application shows that to each admissible triplet l_1, l_2, l there is a complex number, $(l_1, \| M(y)(l) \|, l_2)$, such that for every triplet m_1, m_2, m,

$$(y(l_2, m_2)\ M(y(l,m))y(l_1,m_1)) = c(l_1,l_2,l; m_1,m_2,m)(l_1, \| M(y(l)) \|, l_2).$$

Here, as before, the first factor is a Clebsh-Gordan coefficient and it depends on the algebra $\{A_1, A_2, A_3\}$ only.

Next we need a consequence [1.b] of Racah's version of the Wigner-Eckhart theorem. To describe this consequence we introduce some notations.

In case $\varepsilon(-1)^{l(j)} = 1$, set

$$(4.24)_{+1} \qquad \begin{aligned} &y(\varepsilon,(j+1)j,(l(j)+1)l(j),m) = z(+1,(j+1)j,(l(j)+1)j),m) \\ &y(\varepsilon,(j+1)j,(l(j)-1)l(j),m) = z(-1,(j+1)j,(l(j)-1)l(j),m) \end{aligned}$$

and in case $\varepsilon(-1)^{l(j)} = -1$, set

$$(4.24)_{-1} \qquad \begin{aligned} &y(\varepsilon,(j+1)j,(l(j)+1)l(j),m) = z(-1,(j+1)j,(l(j)+1)l(j),m) \\ &y(\varepsilon,(j+1)j,(l(j)-1)l(j),m) = z(+1,(j+1)j,(l(j)-1)l(j),m). \end{aligned}$$

Then we see from definitions $(4.11)_\pm$ that

$$z(\pm 1,(j+1)j,(l(j)\pm 1)l(j),m)$$
$$\in \mathscr{C}(\pm 1)\otimes E(A_1^2+A_2^2+A_3^2)(l(j)\pm 1)l(j))\cdot \mathfrak{L}_2(\mathscr{S}_2).$$

The algebra generated by $S_k(\pm 1,(l+1)l)$, the part of these operators over these subspaces, is not irreducible. Nevertheless, as is well known [2.b], this algebra is multiplicity free. That is to say, if we decompose this subspace into an orthogonal sum in such a manner that on each summand the algebra is irreducible. then a summand of given dimension occurs exactly once. The multiplicity free character of this algebra allows one to define the Racah coefficients. These coefficients, in turn, allow one to formulate the previously mentioned consequence of Racah's version [1.b] of the Wigner-Eckhart theorem. It says, that equation (4.17) together with the commutator equations (4.18) and (4.20) imply for each half-integer j, j^1 and m, m^1,

$$(z(\pm 1,(j+1)j,(l(j)+1)l(j),m^1),Fz(\pm 1,(j+1)j,(l(j)-1)l(j),m))=\sqrt{\frac{4\pi}{3}}\cdot$$
$$(4.25)\cdot \delta(j,j^1)\delta(m\,.\,m^1)(-1)^{l(j)+1+(\frac{1}{2})-j}.\,W(l(j)-1,\tfrac{1}{2},l(j),\tfrac{1}{2};j,1)\,.$$
$$\cdot\sqrt{2(2l(j)+1)}\cdot(l(j),\|M(y(1))\|,l(j)-1)\cdot(\tfrac{1}{2},\|\tau(\pm 1)(1)\|,\tfrac{1}{2})$$

The last two factors on the right are defined by equations (4.19) and the one after (4.20). The factor $W(\cdots;\cdots)$ is a Racah coefficient in the notation of the book of Rose [1]. As a first consequence of equation (4.21) we see that these matrix elements vanish, unless

$$(4.26)\qquad j=j^1 \quad \text{and} \quad m=m^1.$$

For such values the constants on the right of (4.21) have been determined. Specifically according to [1.f] and [1.d]

$$(\tfrac{1}{2},\|\tau(\pm 1)(1)\|,\tfrac{1}{2})=\frac{\sqrt{3}}{2}$$

and

$$(l,\|M(y(1))\|,l-1)=C(l-1,1,l;0,0)\cdot\sqrt{\frac{3(2l-1)}{4\pi(2+1)}}\,.$$

The Clebsh-Gordan and Racah coefficients corresponding to the Lie algebra of the orthogonal group in $\mathscr{E}_3$ have been tabulated. According to [1.d],

$$C(j-\tfrac{1}{2},j+\tfrac{1}{2};0,0,0)=\sqrt{\frac{j+\frac{1}{2}}{2j}}$$

and

$$W(j-\tfrac{1}{2},\tfrac{1}{2},j+\tfrac{1}{2},\tfrac{1}{2};j,1) = -\sqrt{\frac{1}{2\cdot 3(j+\frac{1}{2})}}\ .$$

Insertion of these equations in (4.25) yields

(4.27) $(z(\pm 1,(j+1)j,(l(j)+1)l(j),m), Fz(\pm 1,(j+1)j,(l(j)-1)l(j),m) = -1,$

if we remember definition (3.33). Next we maintain that for each positive half-integer j and m in $[-j,+j]$,

(4.28) $(z(\pm 1,(j+1)j,(l(j)-1)l(j),m), Fz(\pm 1\ (j+1)j,(l(j)-1)l(j),m) = 0.$

To verify this relation recall definition (4.13), which shows that

$$z(\pm 1,(j+1)j,(l(j)-1)l(j),m) \in \mathscr{C}(\pm 1)\otimes E(A_1^2+A_2^2+A_3^2)((l(j)-1)l(j))\cdot \mathfrak{L}_2(\mathscr{S}_2).$$

Hence this vector is a linear combination of the vectors

$$c(\pm 1,\tfrac{1}{2},m_1)\otimes y(l(j)-1),m_2);\ m_1 = \pm\tfrac{1}{2}, m_2 = -(l(j)-1),\cdots((l(j)-1).$$

Actually the scalars in this linear combination are Clebsh-Gordan coefficients, but we shall not make use of this fact. Specifically,

$$z(\pm 1,(j+1)j,(l(j)-1)l(j),m)$$
$$= \sum_{m_1+m_2=m} C(\tfrac{1}{2},l(j),j;m_1,m_2,m)c(\pm 1,\tfrac{1}{2},m_1)\otimes y((l(j)-1),m_2).$$

Next set

$$\alpha(m_1',m_2',m_1,m_2,k) = (c(\pm,\tfrac{1}{2},m_1'),\tau(1,k)c(\pm,\tfrac{1}{2}\ m_2')).$$
$$\cdot C(\tfrac{1}{2},l(j)-1,j;m_1',m_2',m)\cdot C(\tfrac{1}{2},l(j)-1,j;m_1,m_2,m).$$

Then we see from equation (4.17) that

$$(z(\pm 1,(j+1)j,(l(j)-1)l(j),m), Fz(\pm 1,(j+1)j,(l(j)-1)l(j),m)) = \sqrt{\frac{4\pi}{3}}\cdot$$

(4.29)

$$\cdot\sum_{k=1}^{3}\ \sum_{m_1+m_2=m}\ \sum_{m_1'+m_2'=m} \alpha(m_1,m_2,m_1',m_2',k)(y(l(j)-1,m_2),M(y(1,k))y(l(j)-1,m_2))$$

Relation (3.29) shows that for each positive number l,

$$(y(l,m),M(y(1,k))y(l,m)) = (Qy(l,m),M(y(1,k))Qy(l,m)).$$

Clearly,

$$QM(y(1,k))Q = -M(y(1,k)).$$

Hence

$$(y(l,m), M(y(1,k))y(l,m)) = 0.$$

Insertion of this fact in (4.29) yields the validity of relation (4.28). Finally combining relations (4.28), (4.27) and (4.26) we arrive at the validity of relation $(4.19)_{-1}$ if we remember definitions $(4.24)_{\pm 1}$. Definition (4.16) shows that the operator F is self-adjoint. Inserting this fact in relations (4.28), (4.27) and (4.26) we arrive at the validity of relation $(4.19)_{+1}$. As noted before relations $(4.19)_{\pm 1}$ imply the validity of conclusions $(4.12)_{\pm 1}$. Hence the proof of Lemma 4.2 is complete.

LEMMA 4.3. *The action of the operator C of Lemma* 4.1 *on the basis vectors of definition* (4.11) *is given by*

$$(4.30)_{\pm 1} \quad Cy(\varepsilon,(j+1)j,(l(j)\pm 1)l(j),m) = i(1 \pm l(j))y(\varepsilon,(j+1)j,(l(j)\mp 1)l(j),m).$$

Definition (4.4) together with Lemma 4.2 shows that Lemma 4.3 is implied by

$$(4.31)_{\pm 1} \quad \begin{aligned}&\left(2\sum_{k=1}^{3}\tau_k \otimes A_k\right)y(\varepsilon,(j+1)j,(l(j)\pm 1)l(j),m)\\ &\quad = -(1 \pm l(j))y(\varepsilon,(j+1)j,(l(j)\pm 1)l(j),m).\end{aligned}$$

To verify these relations recall definitions $(3.13)_k$. They show that

$$(4.32) \qquad 2\sum_{k=1}^{3}\tau_k \otimes A_k = \sum_{k=1}^{3} S_k^2 - I \otimes \sum_{k=1}^{3} A_k^2 - \sum_{k=1}^{3}\tau_k^2 \otimes I.$$

According to definition (4.11)

$$\left(\sum_{k=1}^{3} S_k^2\right)y(\varepsilon,(j+1)j,(l(j)\pm 1)l(j),m) = (j+1)jy(\varepsilon,(j+1)j,(l(j)\pm 1)l(j),m).$$

At the same time we see that

$$\begin{aligned}&\left(I \otimes \sum_{k=1}^{3} A_k^2\right)y(\varepsilon,(j+1)j,(l(j)\pm 1)l(j),m)\\ &\quad = (l(j)\pm 1)l(j)y(\varepsilon,(j+1)j,(l(j)\pm 1)l(j),m),\end{aligned}$$

and

$$\left(\sum_{k=1}^{3}\tau_k^2 \otimes I\right)y(\varepsilon,(j+1)j,(l(j)\pm 1)l(j),m) = \tfrac{3}{4}y(\varepsilon,(j+1)j,(l(j)\pm 1)l(j),m).$$

It is an elementary fact that definition (3.33) implies

$$(j+1)j-(l(j)\pm 1)l(j)-\tfrac{3}{4} = -(1+l(j)).$$

Inserting these equations in equation (4.32) we obtain the validity of relations $(4.31)_{\pm 1}$. From these, in turn, we obtain the validity of conclusions $(4.30)_{\pm 1}$. This completes the proof of Lemma 4.3.

Having established these three lemmas we employ them to construct a family of partial isometries which satisfies the conclusions of Theorem 2.2.

To construct such partial isometries recall definitions (3.41) and $(4.11)_{\pm}$. They show that for each triplet (ε, l, m), where ε is ± 1, l is a positive integer and m is a half-integer in $[-j(l), +j(l)]$, we have

$$\begin{aligned}&\mathfrak{E}(\varepsilon, l, m)\otimes\dot{\mathfrak{C}}_{\infty}(0,\infty)\\ &=(-i)y(\varepsilon,(j(l)+1)j(l),(l+(-1)^{l}\varepsilon)l,m)\otimes\dot{\mathfrak{C}}_{\infty}(0,\infty)\oplus\\ &\oplus y(\varepsilon,(j(l)+1)j(l),(l-(-1)^{l}\varepsilon(l,m)\otimes\dot{\mathfrak{C}}_{\infty}(0,\infty).\end{aligned} \tag{4.33}$$

Next set

$$(4.34)_{(\mathfrak{E}^{\perp})} \qquad U(\varepsilon,l,m)\mathfrak{E}(\varepsilon,l,m)^{\perp}\otimes\mathfrak{C}_{\infty}(0,\infty) = 0$$

and

$$(4.34)_{(\mathfrak{E})} \quad U(\varepsilon,l,m)\begin{pmatrix}(-i)y(\varepsilon(-1)^{l},(j(l)+1)j(l),(l+\varepsilon(-1)^{l})l,m)\otimes f\\ +y(\varepsilon(-1)^{l},(j(l)+1)j(l),(l-\varepsilon(-1)^{l})l,m)\otimes g\end{pmatrix} = \begin{bmatrix}f\\ g\end{bmatrix}.$$

It is clear from definition (4.34) that the range of each of these partial isometries is $\mathfrak{L}_2((0,\infty),\mathscr{C}_2)$. That is to say conclusion (2.14) of Theorem 2.2 holds. To verify conclusion (2.15) recall relation (4.33). This shows that the initial set of $U(\varepsilon,l,m)$ is given by

$$\overline{\mathfrak{E}(\varepsilon,l,m)\otimes\dot{\mathfrak{C}}_{\infty}(0,\infty)}.$$

This fact together with the already established conclusion (2.9) of Theorem 2.1 yields the validity of conclusion (2.15) of Theorem 2.2.

The validity of conclusion (2.16) is clear from definition (4.34). In fact it yields the stronger inclusion,

$$U(\varepsilon,l,m)\cdot\mathscr{C}_4\otimes\mathfrak{L}_2(\mathscr{S}_2)\otimes\dot{\mathfrak{C}}_{\infty}(0,\infty)\subset\dot{\mathfrak{C}}_{\infty}((0,\infty),\mathscr{C}_2).$$

To verify conclusion (2.17) recall Lemma 4.1. This lemma together with definition $(2.4)_e$ shows that

$$(I \otimes T)(\dot{H}(e)(I \otimes T)^* = B \otimes \frac{1}{i}(\dot{D} - \dot{M}^{-1}) + C \otimes \dot{M}^{-1}$$

$$(4.35) \qquad + (\gamma_0 \otimes I) \otimes I - e(I \otimes I) \otimes \dot{M}^{-1}$$

$$on\ \mathscr{C}_4 \otimes \mathfrak{C}_\infty(\mathscr{S}_2) \otimes \dot{\mathfrak{C}}_\infty(0, \infty).$$

We see from relation (3.30) and definitions $(4,11)_\pm$ that

$$(\gamma_0 \otimes I)y(\varepsilon, (j(l)+1)j(l), (l \pm \varepsilon 1)^l \varepsilon)l, m) = \pm y(\varepsilon(-1)^l, (j(l)+1)j(l), (l + \pm 1)l, m).$$

Hence from Lemmas 4.3 and 4.2 we obtain

$$Cy(\varepsilon(-1)^l, (j(l)+1)j(l), (l \pm \pm 1)l, m)$$
$$= i(1 \pm \pm 1l)y(\varepsilon(-1)^l, (j(l)+1)j(l), (\mp \pm 1), l, m)$$

and

$$By(\varepsilon(-1)^l (j(l)+1)j(l), (l \pm \pm 1)l, m) = -y(\varepsilon(-1)^l, (j(l)+1)j(l), (l \mp \pm 1)l, m).$$

Inserting these equations in relation (4.35), we arrive at

$$(I \otimes T)H(e)(I \otimes T)^*(-i)y(\varepsilon(-1)^l, (j(l)+1)j(l)(l + \pm)1l, m) \otimes f =$$

$$= y(\varepsilon(-1)^l, (j(l) + 1j(l)(l - \pm 1)l, m) \otimes (\dot{D} + \pm 1l\dot{M}^{-1})f +$$

$$+ (-i)y(\varepsilon(-1)^l, (j(l)+1)j(l), (l + \pm 1)l, m) \otimes (I - e\dot{M}^{-1})f,$$

and

$$(I \otimes T)H(e)(I \otimes T)^* y(\varepsilon(-1)^l, (j(l)+1)j(l), (l - \pm 1)l, m) \otimes g =$$

$$= (-i)y(\varepsilon(-1)^l, (j(l)+1)j(l), (l + \pm 1)l, m) \otimes (-\dot{D} + \pm 1\, l\dot{M}^{-1})g$$

$$- y(\varepsilon(-1)^l, (j(l)+1)j(l), (l - \pm 1)l, m) \otimes (I + e\dot{M}^{-1})g.$$

Finally combining these two equations with definitions (4.34) and (2.13) we arrive at the validity of conclusion (2.17). This completes the proof of Theorem 2.2.

Appendix

Reducing subspaces in terms of spherical harmonics

In this appendix we first choose a particular matrix representation of the Dirac operator. Then we describe in terms of spherical harmonics the complete family of reducing subspaces of Theorem 2.1. This is interesting inasmuch as most of the work on Dirac operators has been done with reference to a particular representation.

To describe such a representation let $\sigma_{1,2,3}$ denote the Pauli spin matrices. That is, set

$$(\mathrm{I.1})_{1,2,3} \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then a set of matrices satisfying the commutator relations $(2.1)_{0,1,2,3}$ is given by

$$(\mathrm{I.2})_{0,1,2,3} \qquad \gamma_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma_k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}, \quad k = 1,2,3.$$

To describe the spherical harmonics, first for each positive integer l let P_l denote the corresponding Legendre polynomial. That is set

$$P_l(x) = \left(\frac{2l+1}{2}\right)^{\frac{1}{2}} \frac{1}{2^l l!} \left(\frac{d}{dx}\right)^l (x^2-1)^l, \qquad -1 < x < +1.$$

Next for each integer n in the closed interval $[0, +l]$ set

$$P_{l,n}(x) = (-1)^n \left[\frac{(l-n)!}{(l+n)!}\right]^{\frac{1}{2}} (1-x^2)^{n/2} \left(\frac{d}{dx}\right)^n P_l(x)$$

and for n in $[-l, 0]$ set

$$P_{l,n}(x) = (-1)^n P_{l,-n}(x).$$

Then the spherical harmonic $y(l,n)$ is defined by

$$(\mathrm{I}-3) \qquad y(l,n)(\theta,\phi) = \frac{1}{\sqrt{2\pi}} e^{in\phi} P_{l,n}(\cos\theta)$$

$$0 \leqq \theta \leqq \pi, \; 0 \leqq \phi < 2\pi.$$

Incidentally note that the ortho-normality relations take the form,

$$\int_0^{\pi}\int_0^{2\pi} y(l,n)(\theta,\phi) y(l',n')(\theta,\phi) \sin\theta \, d\theta \, d\phi = \delta(l,l')\delta(n,n'),$$

although we shall not use this fact.

Next let (ε, j, m) be given, where $\varepsilon = \pm 1$, j is a half-integer and m is a half-integer in $[-j, +j]$. With the aid of the spherical harmonics and the function $l(j)$ of definition (3.33) for each (ε, j, m) we define four vectors in $\mathscr{C}_4 \otimes \mathfrak{L}_2(\mathscr{S}_2)$. Specifically, in case

(I–4) $\varepsilon(-1)^{l(j)} = +1,$

set

$$\text{(I–5)} \quad y\,(\varepsilon,(j+1)j,(l(j)+1)\,l(j),m) = \begin{bmatrix} -\sqrt{j+1-m}\,y(j+\tfrac{1}{2},m-\tfrac{1}{2}) \\ +\sqrt{j+1+m}\,y(j+\tfrac{1}{2},m+\tfrac{1}{2}) \\ 0 \\ 0 \end{bmatrix}$$

and

$$\text{(I.6)} \quad y\,(\varepsilon,(j+1)j,(l(j)-1)l(j),m) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{j+m}\,y(j-\tfrac{1}{2},m-\tfrac{1}{2}) \\ \sqrt{j-m}\,y(j-\tfrac{1}{2},m+\tfrac{1}{2}) \end{bmatrix}.$$

In case

(I.7) $\varepsilon(-1)^{l(j)} = -1$

set

$$\text{(I–8)} \quad y(\varepsilon,(j+1)j,(l(j)+1)l(j),m) = \begin{bmatrix} 0 \\ 0 \\ -\sqrt{j+1-m}\;\; y(j+\tfrac{1}{2},m-\tfrac{1}{2}) \\ \sqrt{j+1+m}\;\; y(j+\tfrac{1}{2},m+\tfrac{1}{2}) \end{bmatrix}$$

and

$$\text{(I.9)} \quad y\,(\varepsilon,\ (j+1)j,(l(j)-1)l(j),m) = \begin{bmatrix} \sqrt{j+m}\,y(j-\tfrac{1}{2},m-\tfrac{1}{2}) \\ \sqrt{j-m}\;\; y(j-\tfrac{1}{2},m+\tfrac{1}{2}) \\ 0 \\ 0 \end{bmatrix}.$$

We maintain that these vectors are parallel to the two unit vectors of definitions $(4.11)_{\pm}$. This is the statement of the lemma that follows.

LEMMA I *For each* (ε,j,m) *the vectors of definitions* (I.5) *or* (I.8) *and* (I.6) *or* (I.9) *are parallel to the vectors of definitions* $(4.11)_{\pm}$ *respectively.*

For brevity we assume that relation (I.4) holds and establish Lemma I for the vector of definition (I.5) only.

To verify

$$\text{(I.10)} \quad S_3 y(\varepsilon,(j+1)j,(l(j)+1)l(j),m) = m y(\varepsilon,(j+1)j,(l(j)+1)l(j),m),$$

note that definitions $(3.4)_k$ in this representation yield

$$\text{(I.11)}_k \qquad \tau = \tfrac{1}{2}\begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix}, \quad k = 1,2,3$$

It is clear from definition $(I.1)_3$ that

(I.12) $$\sigma_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \sigma_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

These two relations together with definitions $(3.13)_3$ and (I.5) yield the validity of (I.10) if we use that [2.g] for each spherical harmonic

(I.13) $$A_3 y(j+\tfrac{1}{2}, m \mp \tfrac{1}{2}) = (m \mp \tfrac{1}{2}) y(j+\tfrac{1}{2}, m \mp \tfrac{1}{2}).$$

To verify

(I.14) $$(\gamma_0 \otimes Q) y(\varepsilon, (j+1)j, (l(j)+1)l(j), m) = \varepsilon y(\varepsilon, (j+1)j, (l(j)+1)l(j), m),$$

note that assumption (I.4) implies that $\varepsilon = (-1)^{l(j)}$. Recall relation (3.29) which says that

$$Q y(j+\tfrac{1}{2}, m \mp \tfrac{1}{2}) = (-1)^{j+\frac{1}{2}} y(j+\tfrac{1}{2}, m \mp \tfrac{1}{2}).$$

These two facts together with definitions (3.23) and (I.5) yield the validity of relation (I.14) if we use $(I.2)_0$.

To verify

(I.15) $$(S_1^2 + S_2^2 + S_3^2) y(\varepsilon, (j+1)j, (l(j)+1)l(j), m) = = (j+1) \cdot y\ \varepsilon, (j+1)j,\ (l(j)+1)l(j), m),$$

recall definitions $(3.13)_k$. They show that

(I.16) $$S_1^2 + S_2^2 + S_3^2 = (\tau_1^2 + \tau_2^2 + \tau_3^2) \otimes I + I \otimes (A_1^2 + A_2^2 + A_3^2) + 2 \sum_{k=1}^{3} \tau_k \otimes A_k.$$

Relations $(I.10)_k$ together with definitions $(I.1)_{1,2,3}$ yield

$$\tau_1^2 + \tau_2^2 + \tau_3^2 = \tfrac{3}{4} I,$$

and is well known [2], for each l and n

(I.17) $$(A_1^2 + A_2^2 + A_3^2) y(l, n) = l(l+1) y(l, n).$$

Hence remembering definition (I.5) we obtain

$$(S_1^2 + S_2^2 + S_3^2) y(\varepsilon, (j+1)j, (l(j)+1)l(j), m) = \{[\tfrac{3}{4} + (j+\tfrac{3}{2})(j+\tfrac{1}{2})]I + + 2 \sum_{k=2}^{3} \tau_k \otimes A_k\} \cdot y(\varepsilon, (j+1)j, (l(j)+1)l(j), m).$$

To evaluate the second term we need that the spherical harmonics form a canonical basis [2.g]. In particular we need that

$$(A_1 \pm iA_2)y(l,n) = \sqrt{(l \mp n)(l \pm n + 1)}\,y(l,n \pm 1).$$

This yields

$$2A_1 y(l,n) = \sqrt{(l-n)(l+n+1)}\,y(l,n+1) + \sqrt{(l+n)(l-n+1)}\,y(l,n-1)$$
$$2iA_2 y(l,n) = \sqrt{(l-n)(l+n+1)}\,y(l,n+1) - \sqrt{(l+n)(l-n+1)}\,y(l,n-1).$$

Combining these two equations with (I.13) we see, after an elementary computation, that for each $(\alpha.\beta)$

$$2\sum \tau_k \otimes A_k \begin{bmatrix} \alpha y(l,n) \\ \beta y(l,n+1) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (\alpha n + \beta\sqrt{(l+n+1)(l-n)}\,y(l,n) \\ (-\beta(n+1) + \alpha\sqrt{(l+n+1)(l-n)}\,y(l,n+1) \\ 0 \\ 0 \end{bmatrix}.$$

In particular,

$$(\text{I.18}) \quad 2\sum \tau_k \otimes A_k \begin{bmatrix} -\sqrt{l-n}\,y(l,n) \\ \sqrt{l+n+1}\,y(l,n+1) \\ 0 \\ 0 \end{bmatrix} = -(l+1)\begin{bmatrix} -\sqrt{l-n}\,y(l,n) \\ \sqrt{l+n+1}\,y(l,n+1) \\ 0 \\ 0 \end{bmatrix}.$$

Set

$$l = l(j) = j + \tfrac{1}{2} \quad \text{and} \quad n = m - \tfrac{1}{2}$$

in this relation and recall (I.17). Insertion of these relations in equation (I.16) yields the validity of equation (I.15), if we use that

$$\tfrac{3}{4} + l(l+1) - (l+1) = (l - \tfrac{1}{2})(l + \tfrac{1}{2}) = (j+1)j.$$

Finally we combine equations (I.10), (I.13), (I.15) and recall that according to relations $(3.32)_l$ the intersection of these eigenspaces is one dimensional. These facts together show that the vector of definition (I.5) is parallel to the vector of definition $(4.11)_+$ or $(4.11)_-$. A repetition of these arguments that we shall not carry out, shows that the vector of definition (I.6) is also parallel to either the vector of definition $(4.11)_+$ or to the vector of definition $(4.11)_-$. This completes the proof of Lemma I.

Having established this lemma we can easily describe the reducing subspaces of Theorem 2.1. To do this first recall definition (3.33) which says that

$$j(l) = l - \tfrac{1}{2}, \; l = 1, 2, \cdots.$$

Then define four $\mathscr{C}_4$-valued functions of the polar angles (θ, ϕ) by setting

$$(\text{I.19})_1 \quad w_1(l,m)(\theta,\phi) = \begin{bmatrix} -\sqrt{j(l)+1-m}\; y(j(l)+\tfrac{1}{2}), m-\tfrac{1}{2})(\theta,\phi) \\ \sqrt{j(l)+1-m}\; y(j(l)+\tfrac{1}{2}), m+\tfrac{1}{2})(\theta,\phi) \\ 0 \\ 0 \end{bmatrix}$$

and

$$(\text{I.19})_2 \quad w_2(l,m)(\theta,\phi) = \begin{bmatrix} \sqrt{j(l)+m}\; y(j(l)-\tfrac{1}{2}, m-\tfrac{1}{2})(\theta,\phi) \\ \sqrt{j(l)-m}\; y(j(l)-\tfrac{1}{2}, m+\tfrac{1}{2})(\theta,\phi) \\ 0 \\ 0 \end{bmatrix}$$

and

$$(\text{I.19})_3 \quad w_3(l,m)(\theta,\phi) = \begin{bmatrix} 0 \\ 0 \\ -\sqrt{j(l+1-m}\; y(j(l)+\tfrac{1}{2}, m-\tfrac{1}{2})(\theta,\phi) \\ \sqrt{j(l)+1+m}\; y(j(l)+\tfrac{1}{2}, m+\tfrac{1}{2})(\theta,\phi) \end{bmatrix},$$

and

$$(\text{I.19})_4 \quad w_4(l,m)(\theta,\phi) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{j(l)+m}\; y(j(l)-\tfrac{1}{2}, m-\tfrac{1}{2})(\theta,\phi) \\ \sqrt{j(l)-m}\; y(j(l)-\tfrac{1}{2}, m+\tfrac{1}{2})(\theta,\phi) \end{bmatrix}$$

Combining these definitions with Lemma I and relation (4.33) we obtain that, for $\varepsilon(-1)^l = +1$,

$$(\text{I.20}) \quad \mathfrak{C}(\varepsilon,l,m)\otimes\dot{\mathfrak{C}}_\infty(0,\infty) = \{f(r)w_1(l,m)(\theta,\phi) + g(r)w_4(l,m)(\theta,\phi)\},$$

and for $\varepsilon(-1)^l = -1$

$$(\text{I.21}) \quad \mathfrak{C}(\varepsilon.\, l,m)\otimes\dot{\mathfrak{C}}_\infty(0,\infty) = [(fr)w_2(l,m)(\theta,\phi) + g(r)w_3(l,m)(\theta,\phi)],$$

where in both cases,

$$f, g \in \dot{\mathfrak{C}}_\infty(0,\infty)\ .$$

Note that according to Theorem 2.1 these are reducing subspaces for the operator $(I\otimes T)\dot{H}(e)(I\otimes T)^*$.

To obtain reducing subspaces for the operator $\dot{H}(e)$ set for $x\in\mathscr{E}_3$,

$$\tilde{w}_q(l,m)(x) = w_q(l,m)(\theta,\phi), \qquad q = 1,2,3,4.$$

where

$$x = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \ \theta\in[0,\pi], \phi\in[0,2\pi).$$

Then remembering definition (2.4) we see from relations (I.20) and (I.21) that for $\varepsilon(-1)^l = 1$

(I.22) $(I \otimes T^*)(\mathfrak{E}(\varepsilon,l,m)\otimes\mathfrak{C}_\infty(0,\infty))$

$$= \left\{\frac{1}{|x|}f(|x|)\tilde{w}_1(l,m)(x) + \frac{1}{|x|}g(|x|\tilde{w}_4(l,m)(x)\right\}$$

and for $\varepsilon(-1)^l = -1$

(I.23) $(I\otimes T^*)(\mathfrak{E}(\varepsilon,l,m)\otimes\mathfrak{C}_\infty(0,\infty))$

$$= \left\{\frac{1}{|x|}f(|x|)\tilde{w}_2(l,m)(x) + \frac{1}{|x|}g(x)\tilde{w}_3(l,m)(x)\right\},$$

where in both cases

$$f,g, \in\dot{\mathfrak{C}}_\infty(0,\infty).$$

In other words we hawe expressed the complete family of reducing subspaces of Theorem 2.1 in terms of spherical harmonics.

References

1. M. E. Rose, *Elementary Theory of Angular Momentum*, Wiley, 1957. (a. Section 19, The Wigner-Eckhart Theorem; b. Formula (6.21) in Section 6.24; c. Equation (5.19b) in Section 19; d. Tables I.2 and I.4 in Appendix I; e. Formula (III.22) in Appendix III; f. Page 154).

2. I. M. Gelfand, R. A. Minlos and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications*, Pergamon Press, 1963. (a. Section 2.3; b. Theorems at the end of Section 4.3 together with Section 2.3; c. Equations (7), (8) and (9) in Section 3.2; d. Section 2.1; e. Section 3.1; f. Section 3.3; g. Equation (18) in Section 2.3)

3. A. Messiah, *Mécanique Quantique*, Volume 2, Dunod, Paris, 1960. (a. Equations (79) in Section XIII.19; b. Théoreme Fondamental in Section XX.10; c. Equation (156) in Section XX.25 together with equation (49) in Section XX.9; d. Equation (118) in Section XX.18; e. End of Section XIII.10; f. Equation (116) in Section XX.18).

4. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966. (a. Section VI.5.3; b. Section V. 5.3; c. Section V.5.4; d. Sections III.5.6 and V.3.9).

5. W. Miller, Jr., *Lie Theory and Special Functions*, Academic Press, 1968.

6. P. R. Halmos, *Introduction to Hilbert Space*. (See Theorem 29.1)

7. E. Wigner, *On the matrices which reduce the Kronecker products of representations of S. R. groups*, pp. 87–133, In Quantum Theory of Angular Momentum, Biedenharn and Van Dam, Editors. (See Equation 67 in Section 13).

8. A. Galindo, *Extension del teorema de Wigner-Eckhart*, Anales de la Real Sociedad Española de Fisica y Quimica, Serie A — Fisica. Septiembre-Octubre, pp. 293–297, 1967.

9. A. Galindo, *Topics in advanced quantum mechanics and field theory*, pp. 1–128. In The Proceedings of the 1968 CERN School of Physics. CERN Report 23–68. (See Appendix 2).

10. Gil de Lamadrid, Jesús, *Lectures on group representations*, University of Minnesota, 1967–1968.

11. J. D. Pincus, *Lectures on spectral theory*, State University of New York at Stony Brook, 1967–1968.

12. M. H. Stone, *Linear Transformations in Hilbert Space*. Amer. Math. Soc. Coll. XV, 1932. (See Definition IV.5).

13. F. H. Brownell, *Second quantization and recalibration of the Dirac Hamiltonian of a single electron atom without radiation*, J. D' Anal. Math. **16** (1966), 1–422.

14. A. Brown, and C. Pearcy, *Spectra of tensor products of operators*, Proc. Amer. Math. Soc. **17** (1966), 162–169.

15. A. T. Dash, and M. Schechter, *Tensor products and joint spectra*, Israel J. Math. **8** (1970), 191–193.

16. Courant–Hilbert, *Methods of Mathematical Physics*, Vol. I., Wiley., N. Y. 1953. See Section IV.3.6.

17. J. Dixmier, *Les Algebres D'operateurs Dans L'espace Hilbertien*, Vol I., Gauthier-Villars, Paris, 1957. See Proposition II.2.9. and its Corollary.

18. R. Hermann, *Lie Groups For Physicists*. Benjamin, N.Y., 1966. See Chapter 12.

UNIVERSITY OF MINNESOTA
SCHOOL OF MATHEMATICS
MINNEAPOLIS, MINN. 55455